

# Characterizing the Entangling Capacity of $n$ -qubit Computations

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## Key Ideas

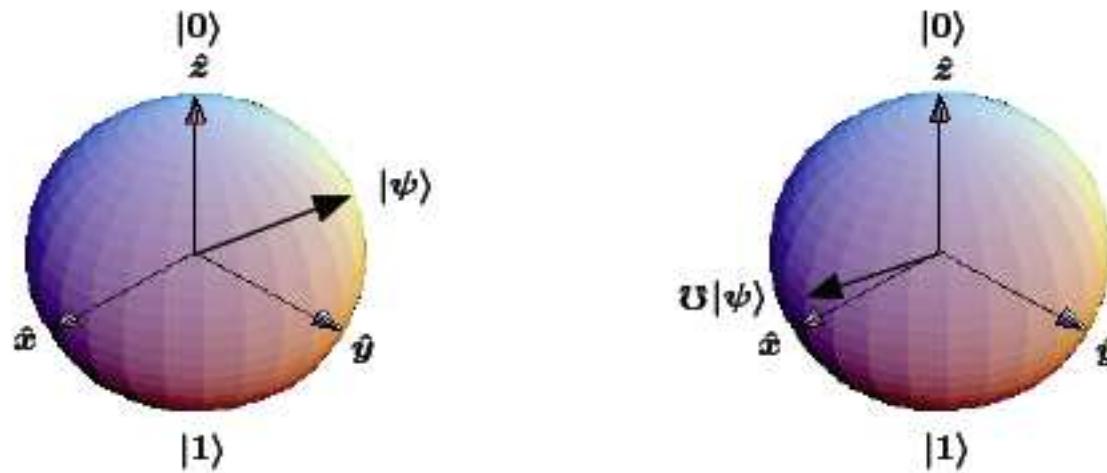
- Tools for studying entanglement (**concurrence**) dynamics
  - dynamics: unitary operator acting on **all of  $\mathcal{H}_n$**
  - **trade-off**: blunt instrument due to accounting for all  $2^n$  dimensions of inputs
- Examples of quantum computation  $\implies$  **structured** entanglement dynamics
- Many open questions  $\implies$  **read paper**

# **Outline**

- I. Concurrence dynamics of Grover search
- II. Concurrence canonical decomposition
- III. Concurrence capacities with examples
- IV. Open questions

## Quantum Bit-Flip $\mathcal{U}$ : One Qubit

- Picture: reverse **Bloch sphere** vector
- Bloch sphere:  $\{ |\psi\rangle ; |\psi\rangle \in \mathcal{H}_1 \} \in \mathbb{CP}^1 = \mathbb{C}^2 / (v \sim r e^{it} v)$



# Quantum Bit-Flip $\mathcal{U}$ & Concurrence

- One qubit formula:  $|\psi\rangle \xrightarrow{\mathcal{U}} (-i\sigma^y)\overline{|\psi\rangle}$
- $\mathbb{C}$ -antilinear: **not unitary evolution**
- $n$ -qubit formula:  $|\psi\rangle \xrightarrow{\mathcal{U}} (-i\sigma^y)^{\otimes n}\overline{|\psi\rangle} = \overline{(-i\sigma^y)^{\otimes n}|\psi\rangle}$
- Physical interpretation: time-reversal symmetry operator
- **Concurrence Entanglement Monotone:**  $C_{2p}(|\psi\rangle) = |\langle\psi|\mathcal{U}|\psi\rangle|/\langle\psi|\psi\rangle$

## Concurrence Entanglement Monotone: Examples

- $C_{2p-1}(|\psi\rangle) = 0$ , for any  $|\psi\rangle$
- $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ , then  $C_4(|\text{GHZ}\rangle) = 1$
- $|\text{W}\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$ , then  $C_4(|\text{W}\rangle) = 0$
- $C_{2p}(|\psi_{2p-1}\rangle \otimes |\psi_1\rangle) = 0$ , in particular vanishes on locals

## Review: Grover Search Algorithm

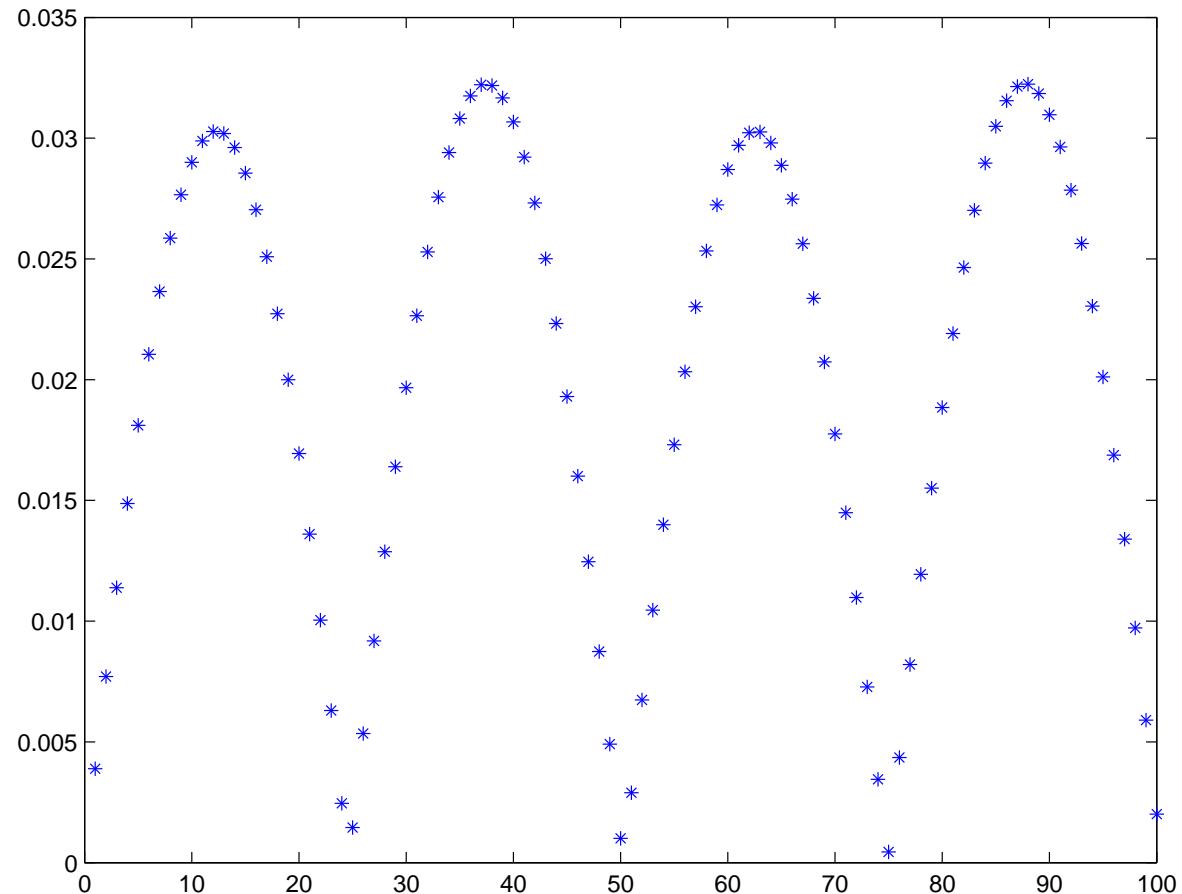
- $n$ -qubits: states  $|k\rangle$ ,  $0 \leq k \leq N - 1$ ,  $N = 2^n$ , w/ search target state  $|x\rangle$
- Quantum oracle: unitary  $O$ ,  $O_x|x\rangle = -|x\rangle$ ;  $O_x|j\rangle = |j\rangle$  else
- Grover's algorithm:
  - build Grover computation  $G = H^{\otimes n} (I_N - |0\rangle\langle 0|) H^{\otimes n} O$
  - compute Grover iterates  $|\psi^k\rangle = G^k(H^{\otimes n}|0\rangle)$
  - $|\psi^\ell\rangle$  has large  $|x\rangle$  amplitude,  $\ell \approx \frac{\cos^{-1} N^{-1/2}}{2\cos^{-1}[(1-\frac{1}{N})^{1/2}]} \in O(\sqrt{N})$

## Concurrence of Grover Iterates

- Simplification:  $x = 11 \cdots 1 = (N - 1)_{\text{bin}}$
- $|\alpha\rangle = \frac{1}{\sqrt{N-1}} \sum_{j=0}^{N-2} |j\rangle$ ,  $|\beta\rangle = |x\rangle$ ,  $\theta = 2 \cos^{-1}[(1 - \frac{1}{N})^{1/2}]$   
$$|\psi^k\rangle = \cos\left(\frac{2k+1}{2}\theta\right) |\alpha\rangle + \sin\left(\frac{2k+1}{2}\theta\right) |\beta\rangle$$
- Computations  $\Rightarrow C_{2p}(|\psi^k\rangle) = \left| \sin[(2k+1)\theta] \frac{1}{\sqrt{N-1}} - \cos^2\left(\frac{2k+1}{2}\theta\right) \frac{2}{N-1} \right|$
- Oddity: let  $Q(-)$  denote Meyers'  $Q$ -measure

$$Q(|\psi^k\rangle) = \left( \frac{N}{2} - 1 \right) \left[ C_{2p}(|\psi^k\rangle) \right]^2$$

## Plot: $C_{10}(|\psi^k\rangle)$ against $k$



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## Natural: CCD by $G = KAK$ metadecomposition

- Metadecomposition: theorem outputting matrix decompositions
- Three cascading inputs, each dependent on last
- Generalizes canonical dec.:  $SU(4) = [SU(2) \otimes SU(2)]\Delta[SU(2) \otimes SU(2)]$ 
  - $SU(2) \otimes SU(2)$ : two-qubit local unitary (LU) group
  - $\Delta$ : relative phase computations on Bell (or “magic”) basis
- Sample applications: 2q control theory, 2q quantum logic circuits, 2q computation times, 2q entanglement theory

## Cascading Inputs of $G = KAK$ theorem

1. Lie group  $G$ , semisimple,  $\mathfrak{g} = \text{Lie}(G)$  ( $\implies G = \exp \mathfrak{g}$ , matrix exponential)
2. Cartan involution  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$  (using same term if  $G$  compact)
  - $[\theta X, \theta Y] = \theta[X, Y]$
  - $\theta^2 = I_N$  (involution)
  - Notation:  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ , where  $\mathfrak{p} = \mathfrak{g}_{-1}$ ,  $\mathfrak{k} = \mathfrak{g}_{+1}$
  - Encodes generalized polar decomp. of  $\mathfrak{g}$
3. (Maximal-Commutative) subspace  $\mathfrak{a} \subset \mathfrak{p}$

## $G = KAK$ Example: Bloch Sphere Rotations

- Take  $G = SU(2)$  (prefix  $SG$ : determinant-one subgroup)
- $\mathfrak{su}(2) = \{A = iH \in \mathbb{C}^{2 \times 2} ; A^\dagger = -A, \text{tr } A = 0\}$
- $\theta(X) = -X^T = \overline{X}$ , fixes  $\mathfrak{k} = \mathfrak{so}(2) = \mathfrak{su}(2) \cap \mathbb{R}^{2 \times 2}$
- $\{R_y(t)\} = SO(2) = \exp \mathfrak{so}(2)$ , the  $Y$ -axis Bloch sphere rotations
- Take  $\mathfrak{a} = \mathbb{R}(i|0\rangle\langle 0| - i|1\rangle\langle 1|)$ , so  $A = \{R_z(t)\}$
- Result:  $SU(2) = \{R_y(t)\}\{R_z(t)\}\{R_y(t)\}$ , the  **$YZY$ -decomposition**

## $G = KAK$ Example: 2q Canonical Decomposition

- $G = SU(4)$ , with  $\mathfrak{g} = \mathfrak{su}(4) = \{X = iH \in \mathbb{C}^{4 \times 4} ; H = H^\dagger, \text{tr } H = 0\}$
- $\theta(X) = (-i\sigma^y)^{\otimes 2}[-X^T](-i\sigma^y)^{\otimes 2} = (-i\sigma^y)^{\otimes 2}[\bar{X}](-i\sigma^y)^{\otimes 2}$
- Check: for this  $\theta$ , in fact  $K = SU(2) \otimes SU(2)$ 
  - +1 eigenspace of  $\theta$  is  $[I_2 \otimes \mathfrak{su}(2)] \oplus [\mathfrak{su}(2) \otimes I_2]$
  - Product Rule:  $\text{Lie}[SU(2) \otimes SU(2)] = [I_2 \otimes \mathfrak{su}(2)] \oplus [\mathfrak{su}(2) \otimes I_2]$
- May choose  $A = \Delta$  phasing Bell basis so that  $\mathfrak{a} \subset \mathfrak{p}$
- Result: canonical dec.  $SU(4) = [SU(2) \otimes SU(2)]\Delta[SU(2) \otimes SU(2)]$

## CCD: Extend $G = KAK$ Inputs

- Extends Euler angle and  $2q$  canonical decomp. examples
- $G = KAK$  Input #1:  $G = SU(2^n)$
- $G = KAK$  Input #2:  $\theta(iH) = [(-i\sigma^y)^{\otimes n}]^\dagger \overline{(iH)} (-i\sigma^y)^{\otimes n} = \mathcal{U}(iH)\mathcal{U}^{-1}$
- $\mathfrak{k}$  is  $+1$ -eigenspace;  $K = \exp(\mathfrak{k})$
- $\mathfrak{a}$ : parity-dependent subalgebra of  $\mathfrak{p}$ ;  $A = \exp(\mathfrak{a})$
- Concurrence Canonical Decomposition is  $SU(N) = KAK$

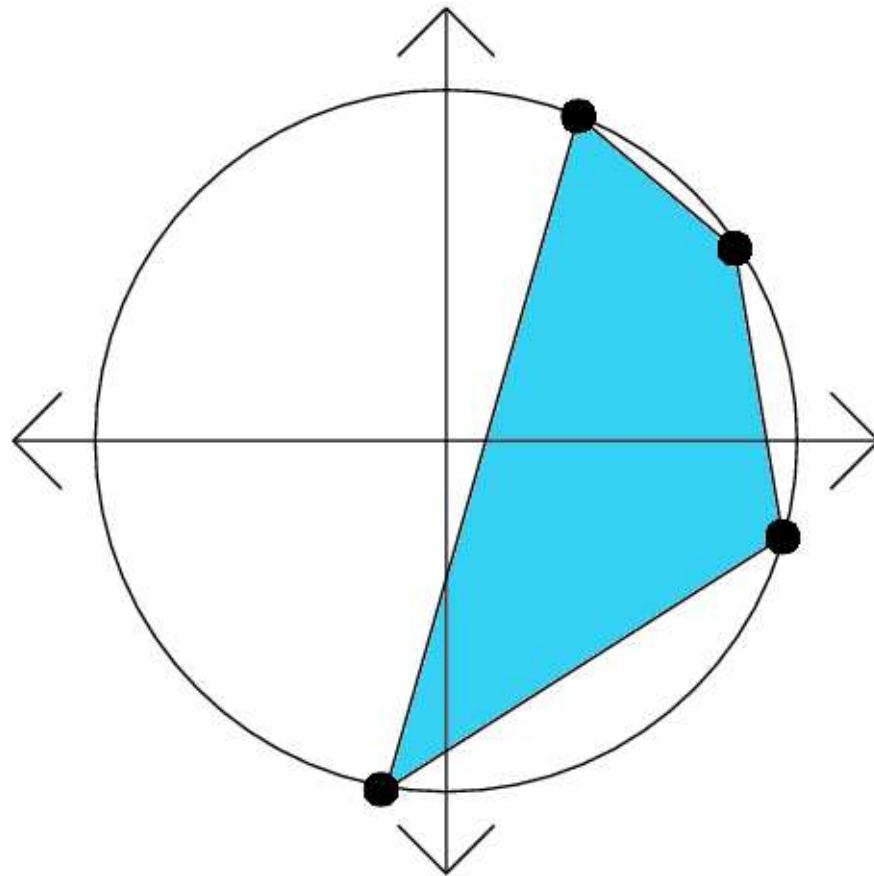
## Two-Qubit Entanglement Capacities

- Zhang et al (Berkeley): two-qubit entanglement capacities
  - Def:  $\mathcal{E}_2(\nu) = \max\{ C_2(\nu|\psi\rangle) ; C_2(|\psi\rangle) = 0 \}$
  - Strategy: study changes in entanglement induced by a factor of two-qubit canonical decomp,  $\nu = [b \otimes c]a[d \otimes f]$
  - Thm:  $\mathcal{E}_2(\nu) = 1$  iff the convex hull (polygonal span) of  $\text{spec}[(\sigma^y)^{\otimes 2} \nu (\sigma^y)^{\otimes 2} \nu^T]$  holds  $0 \in \mathbb{C}$
- Application:  $B$  gate; two-qubit computation with minimal possible (2) applications needed to build  $SU(4)$  from  $\{B, LU\}$

## Even Qubit Concurrence Capacities

- Fix  $n = 2p$ , integral spin qubit system,  $K \cong SO(N)$ 
  - Def: Concurrence capacity
$$\tilde{\kappa}_{2p}(v) = \max\{ C_{2p}(v|\psi\rangle) ; C_{2p}(|\psi\rangle) = 0, \langle\psi|\psi\rangle = 1 \}$$
  - Prop:(BB)  $\tilde{\kappa}_{2p}(k_1ak_2) = \tilde{\kappa}_{2p}(a)$
  - Thm:(BB)  $\tilde{\kappa}_{2p}(v) = 1$  iff the convex hull (polygonal span) of  $\lambda_c(v) = \text{spec}[(-i\sigma^y)^{\otimes 2p} v (-i\sigma^y)^{\otimes 2p} v^T] = \text{spec}(a^2)$  holds  $0 \in \mathbb{C}$
- Application: Choose an element  $a$  at random for  $2p$  large  $\Rightarrow$  the concurrence capacity of  $v = k_1ak_2$  is probably one

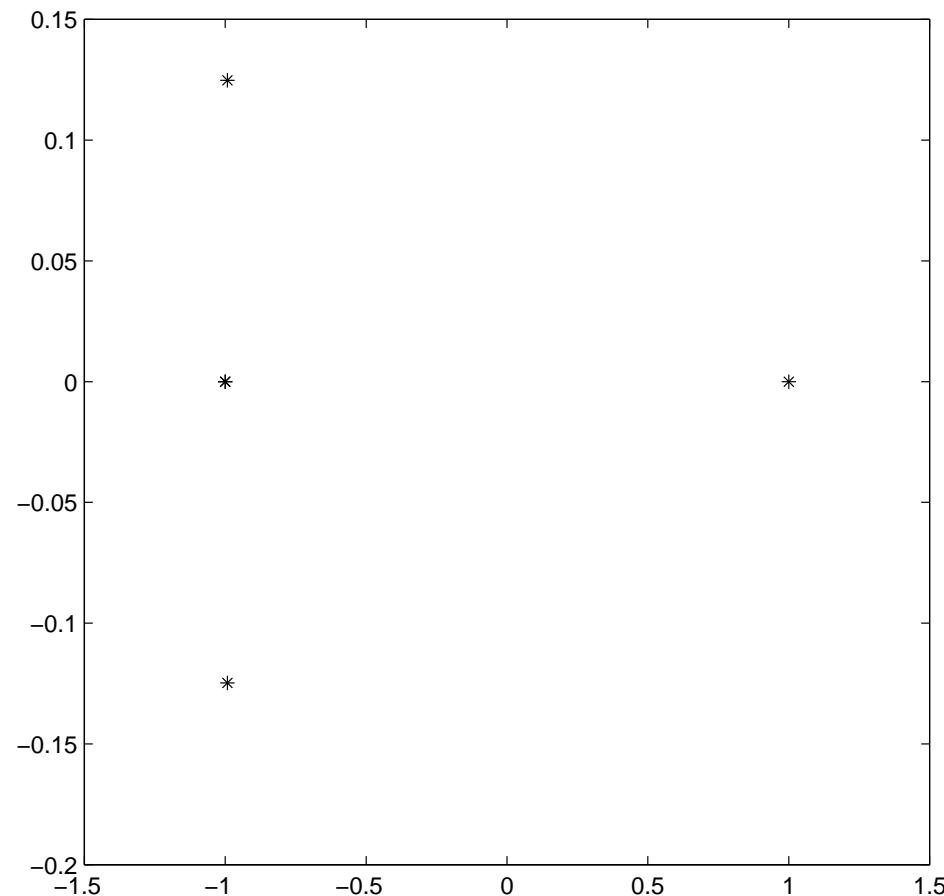
## Picture: Sample Convex Hull



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$\lambda_c(v) = \mathbf{spec}(a^2)$  of 10-qubit Grover  $G$



## $\lambda_c(v) = \mathbf{spec}(a^2)$ of 10-qubit Grover $G$ , Cont.

- Double eigenvalue at  $-1$ , single in quadrants II, III
- 1020-fold degeneracy at  $1$
- Concurrence capacity 1:  $0$  within polygon span
  - Exists  $|\psi\rangle$ ,  $C_{2p}(|\psi\rangle) = 0$ ,  $C_{2p}(G|\psi\rangle) = 1$
  - How entangled is  $|\psi\rangle$ ?

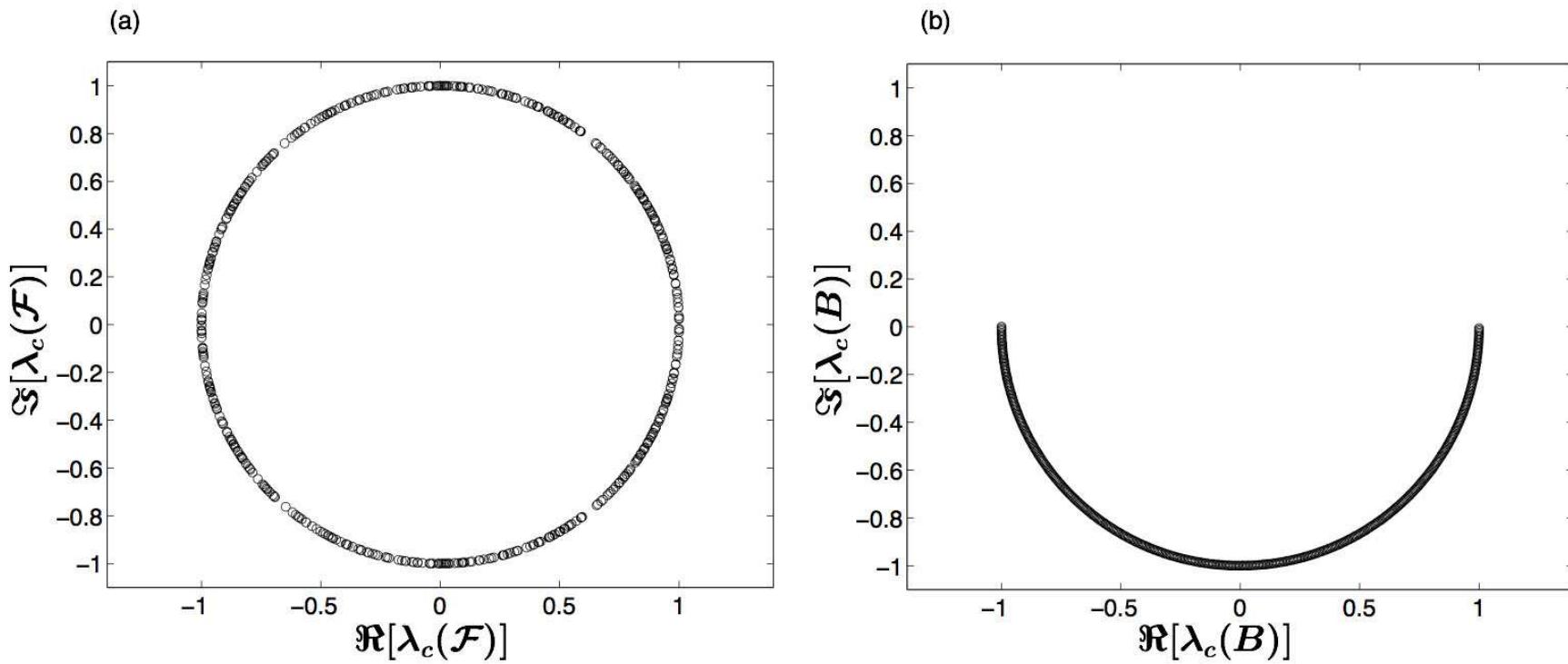
$\mathcal{Q}(-)$  for null-concurrent  $|\psi\rangle$ ,  $C_{2p}(G|\psi\rangle) = 1$

$ \psi\rangle$	choice of weights	$\mathcal{Q}( \psi\rangle)$
$0.500000000000000 0\rangle - 0.03624195588241 1\rangle$ $+0.10560461176979 2\rangle - 0.41308351052014 3\rangle$ $-0.06936265588739 4\rangle + 0.06596124263797 5\rangle$ $-0.24027928828975 6\rangle + 0.24027928828975 9\rangle$ $-0.06596124263797 10\rangle - 0.06936265588739 11\rangle$ $+0.41308351052014 12\rangle + 0.10560461176979 13\rangle$ $-0.03624195588241 14\rangle + 0.50000000000000 15\rangle$	$\frac{1}{2}\lambda_0 + \frac{1}{2}\lambda_4 = 0$	0.982723
$-0.28867513459481 0\rangle - 0.02959143306412 1\rangle$ $+0.08622580444024 2\rangle - 0.62595640857213 3\rangle$ $-0.05663437137612 4\rangle - 0.23481800550717 5\rangle$ $-0.48486235195114 6\rangle - 0.09248791723849 9\rangle$ $-0.34253226368246 10\rangle - 0.05663437137612 11\rangle$ $+0.04860613938250 12\rangle + 0.08622580444024 13\rangle$ $-0.02959143306412 14\rangle + 0.28867513459481 15\rangle$	$\frac{1}{3}\lambda_2 + \frac{1}{3}\lambda_3 + \frac{1}{3}\lambda_4 = 0$	0.789296

## Concurrence Spectra of Other Computations

- Quantum Fourier transform: Fix  $N = 2^n$ ,  $\omega = e^{2\pi i/N}$ 
  - $\mathcal{F} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \omega^{jk} |k\rangle\langle j|$
  - Admits  $\Theta(\log N)$ -sized quantum logic circuit
  - Component of other quantum algorithms, e.g. Shor's factoring
- Quantum baker's map:  $B_n = \mathcal{F}_n(I_2 \otimes \mathcal{F}_{n-1}^{-1})$ 
  - Used in quantum chaos theory; quantum mixing
  - Concurrence of  $B_n^k |00\cdots 0\rangle$  studied by Scott, Caves

$\lambda_c(v) = \mathbf{spec}(a^2)$  of  $\mathcal{F}_{10}$  (**left**) &  $B_{10}$  (**right**)



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## Open Question #1: Pattern in $\lambda_c(B_{2p})$

- Concurrence capacity of quantum baker's map: **not one**
- note that  $-1$  is always an eigenvalue, but not  $1$
- closest to  $1$  is  $\exp(-i\pi/2^{n-1})$ ?

$n$ , # qubits	$\max\{ -i \log(\lambda) ; \lambda \in \lambda_c(B_n) \}$	$\pi/2^{n-1}$
4	-0.392 699 081	0.392 699 081
6	-0.098 174 770	0.098 174 770
8	-0.024 543 693	0.024 543 693
10	-0.006 135 923	0.006 135 923

## Open Question #2: $|\psi\rangle$ realizing capacities

- Grover map  $G = H^{\otimes n}(I - 2|0\rangle\langle 0|)H^{\otimes n}O$ 
  - Produced  $|\psi\rangle$ ,  $C_{2p}(|\psi\rangle) = 0$ ,  $C_{2p}(G|\psi\rangle) = 1$
  - Easiest examples:  $Q(|\psi\rangle)$  large
  - $\lambda_c(v) = \{\lambda_j\}_{j=0}^{N-1}$ : any sum  $0 = \sum_{j=0}^{N-1} t_j \lambda_j \implies$  new  $|\psi\rangle$
  - Produce  $|\psi\rangle$  with  $Q(|\psi\rangle)$  small?
- Replace  $G$  with other maps, e.g.  $\mathcal{F}_{2p}$
- Replace  $Q(-)$  with other entanglement monotones

## Open Question #3: Mean concurrence numbers

- Fact:  $\{ |\psi\rangle ; C_{2p}(|\psi\rangle) = 0, \langle \psi|\psi \rangle = 1 \} = \{ k|00\cdots 0\rangle ; k \in K \}$
- Similar comment for  $C_{2p}^{-1}(\{1\}) = \{ k|\text{GHZ}\rangle ; k \in K \}$
- Define  $\mu_+(v) = \int_K C_{2p}(v k|0\rangle) dk$  and  $\mu_-(v) = \int_K C_{2p}(v^{-1} k|\text{GHZ}\rangle) dk$
- $\mu_\pm(G) < \mu_\pm(\mathcal{F})$ ? Does  $\mu_\pm(-)$  capture more structure of  $\lambda_c(v)$ ?
- More generally, better measure of concurrence change than  $\tilde{\kappa}_{2p}(v)$ ?

## Key Ideas

- Tools for studying entanglement (**concurrence**) dynamics
  - dynamics: unitary operator acting on **all of  $\mathcal{H}_n$**
  - **trade-off**: blunt instrument due to accounting for all  $2^n$  dimensions of inputs
- Examples of quantum computation  $\implies$  **structured** entanglement dynamics
- Many open questions  $\implies$  **read paper**